SEMI-QUANTUM CHAOS

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February 5, 2008

Abstract

We consider a system in which a classical oscillator is interacting with a purely quantum mechanical oscillator, described by the Lagrangian $L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{A}^2 - \frac{1}{2}(m^2 + e^2A^2)x^2$, where A is a classical variable and x is a quantum operator. With $\langle x(t) \rangle = 0$, the relevant variable for the quantum oscillator is $\langle x(t)x(t) \rangle = G(t)$. The classical Hamiltonian dynamics governing the variables $A(t)$, $\Pi_A(t)$, $G(t)$ and $\Pi_G(t)$ is chaotic so that the results of making measurements on the quantum system at later times are sensitive to initial conditions. This system arises as the zero momentum part of the problem of pair production of charged scalar particles by a strong external electric field.

1 Introduction

The definition and observation of chaotic behavior in classical systems is familiar and well understood [1]. However the proper definition of chaos for quantum systems and its experimental manifestations are still unclear [2]. Here we present a simple model of a coupled quantum-classical system and

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introduce a new phenomenon that we call semi-quantum chaos. In a classical chaotic system such as the weather we are accustomed to situations where there is lack of long time forcasting because of the sensitivity of the system to initial conditions. The simple model we present here has the unusual feature that one has to give up long term forcasting even for the quantum mechanical probabilities,as exemplified by the average number of quanta at later times. The complete dynamics of the coupled quantum and classical oscillators is described by a classical effective Hamiltonian that is the expectation value of the quantum Hamiltonian. This effective Hamiltonian displays chaotic behavior, and thus the parameters that describe the quantum mechanical wave function (and hence expectation values) are sensitive to initial conditions. Chaos in dynamical systems with both quantum and classical degrees of freedom has been noted in more complicated systems and in a different context by other authors, (see *e.g.* [3]).

We consider a system in which a classical oscillator is interacting with a purely quantum mechanical oscillator described by the Lagrangian,

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{A}^2 - \frac{1}{2}(m^2 + e^2 A^2)x^2 \,, \tag{1}$$

with equations of motion given by

$$\begin{aligned} \ddot{x} + (m^2 + e^2 A^2)x &= 0 && (2) \\ \ddot{A} + e^2 x^2 A &= 0 \,. && (3) \end{aligned}$$

The Hamiltonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\Pi_A^2 + \frac{1}{2}(m^2 + e^2 A^2)x^2 \,, \tag{4}$$

where $p(t) = \dot{x}(t)$ and $\Pi_A = \dot{A}(t)$. We take $x(t)$ to be a quantum operator and $A(t)$ to be the amplitude of the classical oscillator. We require $[x(t), p(t)] = i$. We now introduce time-independent Heisenberg representation creation and destruction operators, a and $a^\dagger$, by the Ansatz

$$x(t) = f(t)\, a + f^*(t)\, a^\dagger \,, \tag{5}$$

and we note that if $f(t)$ satisfies the Wronskian condition

$$i[f^*(t)\dot{f}(t) - \dot{f}^*(t)f(t)] = 1 \,, \tag{6}$$

then a and $a^\dagger$ satisfies the relation $[a, a^\dagger] = 1$. From (2) and (5), we find that $f(t)$ satisfies the equation of motion

$$\ddot{f} + (m^2 + e^2 A^2) f = 0 \,, \tag{7}$$

with the normalization fixed by the Wronskian condition (6). We can satisfy these two equations by the substitution

$$f(t) = \exp\left[-i \int_0^t \Omega(t')\mathrm{d}t'\right] / \sqrt{2\Omega(t)} \,,$$

where $\Omega(t)$ satisfies the nonlinear differential equation

$$\frac{1}{2}\left(\frac{\ddot{\Omega}}{\Omega}\right) - \frac{3}{4}\left(\frac{\dot{\Omega}}{\Omega}\right)^2 + \Omega^2 = \omega^2 \,, \tag{8}$$

with

$$\omega^2(t) \equiv m^2 + e^2 A^2(t) \,. \tag{9}$$

Now, we choose the initial state vector at $t = 0$ to be the ground state of the operator $\hat{n} = a^\dagger a$, $|\Psi(0)\rangle = |0\rangle$, where $a|0\rangle = 0$. Then, from (5), the average (classical) value of $x(t)$ and $p(t)$ is 0 for all time, $\langle x(t)\rangle = 0$ and $\langle p(t)\rangle = 0$. However, the quantum fluctuations of $x(t)$ are non-zero and are given by the variable $G(t)$,

$$G(t) = \langle x^2(t)\rangle = |f(t)|^2 = \frac{1}{2\Omega(t)} \,. \tag{10}$$

Then, from (8), it is easy to show that $G(t)$ satisfies

$$\frac{1}{2}\left(\frac{\ddot{G}}{G}\right) - \frac{1}{4}\left(\frac{\dot{G}}{G}\right)^2 - \frac{1}{4G^2} + \omega^2 = 0 \,. \tag{11}$$

In addition, we find that

$$\langle \dot{x}^2(t)\rangle = \frac{\dot{G}^2}{4G} + \frac{1}{4G}. \tag{12}$$

The expectation value of Eq. (4) becomes a new effective Hamiltonian

$$\begin{aligned} H_{\text{eff}} &= \langle H(t)\rangle \\ &= \frac{\Pi_A^2}{2} + 2\Pi_G^2 G + \frac{1}{8G} + \frac{1}{2}(m^2 + e^2 A^2) G \,. \end{aligned} \tag{13}$$

The conjugate momenta are

$$\Pi_G = \frac{\dot{G}}{4G}\,, \qquad \Pi_A = \dot{A}\,, \tag{14}$$

This *classical* Hamiltonian determines the variables, G and $\dot{G}$, necessary for a complete *quantum-mechanical* description of this system. Hamilton's equations then yield

$$\begin{aligned} \dot{\Pi}_G &= -2\Pi_G^2 + \frac{1}{8G^2} - \frac{1}{2}\omega^2 \\ \dot{\Pi}_A &= -e^2 A G \end{aligned} \tag{15}$$

or equivalently:

$$\begin{aligned} \ddot{A} + e^2 G\, A &= 0 \\ \frac{1}{2}\left(\frac{\ddot{G}}{G}\right) - \frac{1}{4}\left(\frac{\dot{G}}{G}\right)^2 - \frac{1}{4G^2} + \omega^2 &= 0\,, \end{aligned} \tag{16}$$

which correspond to (11) and the expectation values of Eq. (3).

The classical effective Lagrangian is

$$L_{\text{eff}} = \frac{\dot{A}^2}{2} + \frac{\dot{G}^2}{8G} - \frac{1}{8G} - \frac{1}{2}(m^2 + e^2 A^2) G\,. \tag{17}$$

This Lagrangian could also have been obtained using Dirac's action,

$$\Gamma = \int dt \langle \Psi(t) | i\frac{\partial}{\partial t} - H | \Psi(t) \rangle \equiv \int dt\, L_{\text{eff}}, \tag{18}$$

and a time-dependent Gaussian trial wave function as described in [4]. This variational method was used to study the quantum Henon-Heiles problem in a mean-field approximation [5]. The Gaussian trial wave function is parametrized as follows

$$\Psi(t) = [2\pi G(t)]^{-1/4} \exp[-(x - q(t))^2 (G^{-1}(t)/4 - i\Pi_G(t)) + ip(t)(x - q(t))].$$

We see that $G(t)$ and $\Pi_G(t)$ are the time dependent real and imaginary parts of the width of the wave function. One can prove for our problem that if the quantum oscillator starts at $t = 0$ as a Gaussian, it is described at all times by

the above expression, where $G(t)$ and $\Pi_G(t)$ are totally determined by solving the effective Hamiltonian dynamics. (For our special initial conditions $p(t) = q(t) = 0$). Thus we find that our effective Hamiltonian totally determines the time evolution of the quantum oscillator. One interesting "classical" variable is the expectation value of the time dependent adiabatic number operator, which corresponds to the number of quanta in a situation where the classical A field is changing slowly (adiabatically). For the related field theory problem (see Section 3) of pair production of charged pairs by strong electric fields, this corresponds to the time dependent single particle distribution function of secondaries. To find the expression for the number of quanta, we begin with the wave function corresponding to a slowly varying classical field A:

$$g(t) = \exp\Big[-i\int_0^t \omega(t')\mathrm{d}t'\Big]\Big/\sqrt{2\omega(t)},$$

in terms of which we can decompose the quantum operator via

$$x(t) = g(t)\,b(t) + g^*(t)\,b^\dagger(t)\,. \tag{19}$$

Requiring the momentum operator to have the form

$$p(t) = \dot{x}(t) = \dot{g}(t)\,b(t) + \dot{g}^*(t)\,b^\dagger(t)$$

by imposing $g(t)\dot{b}(t) + g^*(t)\dot{b}^\dagger(t) = 0$, and recognizing that $g(t)$ and $g^*(t)$ satisfy the Wronskian condition by construction, then $b(t)$ and $b^\dagger(t)$ have the usual interpretation as creation and annihilation operators. That is, $[x(t), p(t)] = i$ and $[b(t), b^\dagger(t)] = 1$. Also

$$b(t) = i[g^*(t)\,\dot{x}(t) - \dot{g}^*(t)\,x(t)]\,.$$

$b^\dagger(t)b(t)$ can be interpreted as a *time-dependent* number operator for a slowly varying (adiabatic) classical field A. The time independent basis and the time dependent basis are both complete sets and are related by a unitary Bogoliubov transformation, $b(t) = \alpha(t)\,a + \beta(t)\,a^\dagger$, where

$$\begin{aligned}\alpha(t) &= i[g^*(t)\dot{f}(t) - \dot{g}^*(t)f(t)]\\ \beta(t) &= i[g^*(t)\dot{f}^*(t) - \dot{g}^*(t)f^*(t)]\,,\end{aligned}$$

and where $|\alpha(t)|^2 - |\beta(t)|^2 = 1$. If we choose for initial conditions, $\Omega(0) = \omega(0)$, $\dot{\Omega}(0) = \dot{\omega}(0)$, then one finds that $\alpha(0) = 1$ and $\beta(0) = 0$. These are the

initial conditions appropriate to the field theory problem of pair production. The average value of the time-dependent occupation number is given by

$$n(t) = \langle\, b^\dagger(t) b(t)\,\rangle = |\beta(t)|^2 = (4\Omega\omega)^{-1}\left[(\Omega-\omega)^2 + \frac{1}{4}\left(\frac{\dot\Omega}{\Omega} - \frac{\dot\omega}{\omega}\right)^2\right] \,. \tag{20}$$

Eq. (20) allows us to compute the average occupation number of the system as a function of time.

2 Solutions of the classical equations

We first scale out the mass by letting $t \to m^{-1}\,t$, $A \to m^{-1/2}\,A$, $G \to m^{-1}\,G$, and $e \to e\,m^{3/2}$. Then the scaled equations of motion are

$$\begin{aligned} \ddot A + e^2 G\, A &= 0 \\ \frac{1}{2}\left(\frac{\ddot G}{G}\right) - \frac{1}{4}\left(\frac{\dot G}{G}\right)^2 - \frac{1}{4G^2} + 1 + e^2 A^2 &= 0\,. \end{aligned} \tag{21}$$

In order to explore the degree of chaos as a function of (scaled) energy and coupling parameter e, we calculated surfaces of section and Lyapunov exponents. The surface of section is a slice through the three-dimensional energy shell [1]. That is, for a fixed energy and coupling parameter the points on the surface of section are generated as the trajectory pierces a fixed place (e.g. $A = 0$) in a fixed direction. The hallmark of regular motion is the cross section of a KAM torus which is seen as a closed curve in the surface of section. The hallmark of chaotic motion is the lack of any such pattern in the surface of section. In Fig. 1 we show a plot of a surface of section at $E = 0.8$ and $e = 1.$ where regular and chaotic regions co-exist.

The Lyapunov exponent provides a more quantitative, objective measure of the degree of chaos. The Lyapunov exponent, λ, gives the rate of exponential divergence of infinitesimally close trajectories [6]. Although there are as many Lyapunov exponents as degrees of freedom, it is common to simply give the largest of these. For regular trajectories $\lambda = 0$; for chaotic trajectories the exponent is positive. We define

$$\vec\eta(t) \equiv \lim_{|\vec\delta|\to 0} \frac{\vec z(\vec z_0 + \vec\delta, t) - \vec z(\vec z_0, t)}{|\vec\delta|}\,, \tag{22}$$

where $\vec{z}(\vec{z}_0, t)$ is a point in phase space at time t with initial position $\vec{z}_0$. Then the time evolution for $\vec{\eta}(t)$ is

$$\dot{\vec{\eta}}(t) = \vec{\eta}(t) \cdot \vec{\nabla} \vec{F} \mid_{\vec{z}(\vec{z}_0,t)} , \tag{23}$$

where

$$\dot{\vec{z}}(t) = \vec{F}(\vec{z}(t), t) \tag{24}$$

are the full equations of motion for the system. The Lyapunov exponent is defined as

$$\lambda \equiv \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\vec{\eta}(t)}{\vec{\eta}(0)} \right| . \tag{25}$$

Appendix A of Ref. [6] provides an explicit algorithm for the calculation of all the Lyapunov exponents. Since we cannot carry out the $t \to \infty$ limit computationally, the regular trajectories are those for which $\lambda(t)$ decreases as $1/t$, while the chaotic trajectories give rise to $\lambda(t)$ that is roughly constant in time, as judged by a linear least-squares fit of $\log[\lambda(t)]$ *vs.* $\log(t)$.

We calculated the Lyapunov exponents for three values of the scaled coupling constant e (0.1, 1.0, 10.0) and for energies from 0.5 to 2.0. $E = 0.5$ is the lowest energy possible, corresponding to the zero point energy of the oscillator; there is no upper limit on E. Fifty initial conditions were chosen at random for each energy bin of width 0.1 and coupling parameter. One relevant quantity to study is the chaotic volume, the fraction of initial conditions with positive definite Lyapunov exponents (corresponding to chaotic behavior). Errors in this quantity arise because of the finite number of initial conditions chosen, and because the distinction between zero and positive exponents cannot be made with certainty at finite times. We found that for $e = 0.1$, more than 95% of trajectories were regular for all energies tested; for $e = 1.0$ and 10.0, there is a steadily increasing fraction of chaotic orbits between $0.5 \leq E \leq 1.25$. For $1.25 \leq E \leq 2.0$, more than 90% of these orbits are chaotic.

3 Interpretation

We may now ask what are the physical ramifications of our results. The system of equations studied here is the $k = 0$ mode contribution to the problem of pair production of charged mesons by a strong electric field [7],

with $E(t) = -\dot{A}(t)$ being the value of the time evolving electric field. For that problem the equation for $G(t)$ gets modified and becomes a function of the momentum k of the normal modes of the charged scalar field. Eq. (9) being replaced by $\omega^2(t) \to \omega_k^2(t) = (k - eA(t))^2 + m^2$.

The semi-classical equation for $A(t)$ becomes

$$\ddot{A} = e \int dk (k - eA(t)) G_k(t) \ , \tag{26}$$

which gets contributions from all modes. This system of equations is discussed in detail in [7]. When we sum over all the k modes, $A(t)$ becomes a smooth function of time and is insensitive to initial data. However the number of particles produced in a narrow bin of momentum between k and $k + dk$ depends only on $G_k(t)$ and $\dot{G}_k(t)$. Only if one does a coarse graining over momentum does one lose this sensitivity to initial data. Thus one should observe, as one counts the number of produced charged particles in a detector and increases the resolution, that the number of counts in a narrow momentum bin becomes a rapidly oscillating function of time whose behavior is chaotic. The expression for the number of particles in a given momentum bin k is

$$n(k,t) = (4\Omega_k \omega_k)^{-1} \left((\Omega_k - \omega_k)^2 + \frac{1}{4} \left(\frac{\dot{\Omega}_k}{\Omega_k} - \frac{\dot{\omega}_k}{\omega_k} \right)^2 \right) , \tag{27}$$

which should be compared with Eq. (20). The chaotic behavior of Eq. (20) is shown in Fig. 2. [Also see Fig. 3 of [7]].

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FIGURE CAPTIONS

Figure 1: A plot of the surface of section for energy $= 0.8$, $e = 1.0$, and $A = 0$. Each symbol represents a different trajectory. The one chaotic region is in the center of the plot.

Figure 2: A plot of the occupation number given by Eq. (20) for energy $= 1.8$, $e = 1.0$, $A(0) = 0$, $\Pi_G(0) = 0$. The solid line is for for $G(0) = 0.5$; the dashed line is for $G(0) = 0.5001$. This plot shows the sensitivity to initial conditions.